

# Reheating and Cosmic String Production

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We compute the string production rate at the end of inflation, using the string spectrum obtained in [8] in a near-de Sitter space. Our result shows that highly excited strings are hardly produced, thus the simple slow-roll inflation alone does not offer a cosmic string production mechanism.

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## 1. Introduction

String theory has been successful in resolving some longstanding problems, such as the existence of a consistent theory of quantum gravity. However, many problems remain unsolved [1]. One of the most important problems is that by far string theory has not made any concrete predictions verifiable by experiments, thus we do not know whether string theory is a realistic physical theory or not. Cosmology may be an important arena to test string theory. For example, any evidence of the existence of topological defects such as cosmic superstrings, can be an important support for string theory. Cosmic strings can have two different origins, the field theoretic one and fundamental string theory. The discovery of fundamental cosmic strings would be a spectacular way to verify string theory [2]. Cosmic strings from string theory are characterized by some properties not shared by GUT cosmic strings [3].

In 1980's, it was generally believed that the perturbative fundamental strings can not become cosmic strings, due to an argument of Witten [4]. Moreover, the tension of a fundamental string is close to the Planck scale, while cosmic strings with such a tension are ruled out by experiments. Recently, research of compactifications in string theory shows that the string tension measured in the four-dimensional Einstein frame can be much smaller, and in some situations the instability problem is evaded. Thus cosmic strings as fundamental strings may indeed exist and can be observed in the future experiments.

The current research interest of the creation of strings at the end of inflation is focused on the investigation of the final results of collision of branes. Cosmic strings are inevitably produced in this process as topological defects. Research along this direction is spearheaded by Polchinski and Tye, and their collaborators [5][6][2].

In this paper, we will study creation of strings in a more traditional fashion, namely, through gravitational pair production in a time-dependent background. Related work has been done by Gubser [7] with an effective field theory viewpoint, and a steepest descent contour method has been developed to estimate the production rate of the strings.

As discussed in [7], in a regime of parameter space where a spacetime description gives a good approximation of string dynamics, the on-shell constraint for a given string state boils down to a differential equation describing an oscillator with a time-dependent frequency. When the quantization of strings is carried out in a particular background, the “frequency”  $\omega(t)$  is determined. The quantization of strings in a de Sitter background was recently done by Li et al. [8], and the spectrum of such “small strings” is obtained. We

will use the method developed in [7] to estimate the total rate of string creation, using the spectrum i.e. the equation of motion of string state obtained in [8], which is different from that of [7]. The original equation of motion derived in [8] is in de Sitter space where the Hubble parameter is a constant, but as we will show that in fact it is also valid at the end of inflation and during reheating, where the Hubble parameter is a function of time instead of a constant (see Appendix A). We will show this in Appendix B.

The main result of our investigation can be summarized as follows. Strings are generally produced gravitationally at the end of inflation and during reheating, and the energy density of strings produced is highly suppressed by an exponential factor multiplied by a power factor. Since the Hubble parameter is much smaller than 1 in the unit of  $\alpha' = 1$ , this energy density is very small. Our estimate is approximate quantitatively, due to the fact that there are some approximations used in deriving the string spectrum derived in [8] and in our analytic method in estimating the string production rate. However, this semi-quantitative result strongly suggests to us the picture that highly excited strings are hardly produced during reheating and the production rate is very small.

The organization of this paper is as follows. In section 2 the density of string states is calculated, based on the spectrum obtained in [8]. In section 3 the creation rate and the energy density of strings are estimated. The final section is devoted to discussions. In Appendix A we solve the Friedmann equation directly to get the Hubble parameter as a function of time at the end of inflation and during reheating. In Appendix B We show that the spectrum and equation of motion of strings obtained in [8] are valid in a general flat FRW background, not only in pure de Sitter space.

## 2. Degeneracy of String States

The spectrum of strings in a near de Sitter background is different from the one in flat spacetime, and depends on two integers (we shall consider the bosonic sector only in this paper), as will be shown in the next section. These integers are eigenvalues of two operators: the number operator  $N$  and the other operator  $L$ , defined respectively as follows

$$\begin{aligned} N &= \sum_{i=1}^d \sum_{m=1}^{\infty} m \left( N_m^i + \tilde{N}_m^i \right) \\ L &= \sum_{i=1}^d \sum_{m=1}^{\infty} \left( N_m^i + \tilde{N}_m^i + 2N_m^i \tilde{N}_m^i \right), \end{aligned} \tag{2.1}$$

where  $d = D - 1$  and  $D$  is the number of dimensions of spacetime,  $N_m^i$  and  $\tilde{N}_m^i$  are the occupation numbers of the left-mover and the right-mover respectively,  $i$  is the space index and  $m$  is the oscillator index. We denote  $n$  as the eigenvalue of  $N$  and  $l$  as the eigenvalue of  $L$ . The degeneracy of states at level  $n$  and fixed  $l$  is denoted by  $D_{n,l}$ , which is encoded in a generating function as the coefficient of  $z^n w^l$

$$Z(z, w) \equiv \text{tr} z^N w^L, \quad (2.2)$$

or

$$Z(z, w) = \sum_{n,l=0}^{\infty} D_{n,l} z^n w^l. \quad (2.3)$$

Using (2.1) and (2.2), the generating function can be evaluated by an elementary method of quantum statistical mechanics as follows

$$Z(z, w) = \prod_{i=1}^d \prod_{m=1}^{\infty} \sum_{N_m^i, \tilde{N}_m^i=0}^{\infty} z^{m(N_m^i + \tilde{N}_m^i)} w^{N_m^i + \tilde{N}_m^i + 2N_m^i \tilde{N}_m^i}. \quad (2.4)$$

Summing over <sup>1</sup>  $\tilde{N}_m^i$

$$\begin{aligned} Z(z, w) &= \prod_{i=1}^d \prod_{m=1}^{\infty} \sum_{N_m^i=0}^{\infty} \frac{z^{mN_m^i} w^{N_m^i}}{1 - z^m w^{2N_m^i+1}} \\ &= \prod_{i=1}^d \prod_{m=1}^{\infty} \sum_{N_m^i=0}^{\infty} \frac{z^{mN_m^i}}{1 - z^m} f(w, N_m^i), \end{aligned} \quad (2.5)$$

where

$$f(w, N_m^i) = \frac{(1 - z^m) w^{N_m^i}}{1 - z^m w^{2N_m^i+1}}. \quad (2.6)$$

If  $w = 1$ , (2.5) is just the usual generating function of the degeneracy of bosonic string states [9]. The effect of  $w$  is to deform this generating function. This deformation is small when it comes to evaluate the coefficient  $D_{n,l}$  by the steepest descent method. We will expand the generating function i.e. (2.6) near  $w = 1$  to the second order, with higher order terms truncated. This assumption is reasonable as we will show in the saddle point calculation.

$$f(w, N_m^i) = 1 + (w - 1) \left[ \frac{z^m}{1 - z^m} + N_m^i \left( \frac{1 + z^m}{1 - z^m} \right) \right] + O((w - 1)^2). \quad (2.7)$$

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<sup>1</sup> Of course one can sum over  $N_m^i$  first.

Thus the summation in (2.5) reads:

$$\sum_{N_m^i=0}^{\infty} \frac{z^m N_m^i}{1-z^m} f(w, N_m^i) \approx \frac{1}{(1-z^m)^2} \left[ 1 + (w-1) \frac{2z^m}{(1-z^m)^2} \right], \quad (2.8)$$

then the generating function is

$$Z(z, w) \approx \left\{ \prod_{m=1}^{\infty} \frac{1}{(1-z^m)^2} \left[ 1 + (w-1) \frac{2z^m}{(1-z^m)^2} \right] \right\}^d. \quad (2.9)$$

Let  $z \equiv \exp(-\frac{1}{T})$  and  $x \equiv \frac{m}{T}$ . Thus  $x$  is continuous when  $T$  is large enough. In our following calculation, around the saddle points,  $z$  is close to 1 thus  $T$  is indeed large. The infinite product in (2.9) is approximated by an exponential of an integral. Taking logarithm of both sides of (2.9) we obtain

$$\begin{aligned} \frac{\ln Z(z, w)}{d} &\approx -2T \int_{\frac{1}{T}}^{\infty} dx \ln(1 - e^{-x}) + T \int_{\frac{1}{T}}^{\infty} dx \ln \left( 1 + (w-1) \frac{2e^{-x}}{(1 - e^{-x})^2} \right) \\ &\approx \frac{\pi^2 T}{3} + 2(w-1)T \int_{\frac{1}{T}}^{\infty} \frac{dx e^{-x}}{(1 - e^{-x})^2} \\ &= -\frac{\pi^2}{3 \ln z} - \frac{2(w-1)z}{(1-z) \ln z}, \end{aligned} \quad (2.10)$$

in the second line we have truncated higher orders of  $w-1$  in the logarithm function.

By definition, the degeneracy of string states is

$$D_{n,l} = \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} \frac{Z(z, w)}{z^{n+1} w^{l+1}}. \quad (2.11)$$

Here  $Z(z, w)$  vanishes rapidly as  $z \rightarrow 1$  when  $w < 1$ , and  $z^{n+1}$  is very small for  $z < 1$  when  $n$  is very large. Consequently, for large  $n$ , there is a sharply defined saddle point for  $z$  near 1. Indeed, the factor <sup>2</sup>

$$\exp \left[ -\frac{2\pi^2}{3 \ln z} - \frac{4(w-1)z}{(1-z) \ln z} - (n+1) \ln z - (l+1) \ln w \right] \quad (2.12)$$

is stationary for

$$\ln z = -\sqrt{\frac{1}{n+1} \left( \frac{2\pi^2}{3} - \frac{8(w-1)}{\ln z} \right)} \sim -\sqrt{\frac{1}{n+1} \frac{2\pi^2}{3}}. \quad (2.13)$$

Therefore one finds that as  $n \rightarrow \infty$

$$D_{n,l} \sim n^{-5/4} \exp \left( \frac{2\pi}{3} \sqrt{6n} \right) \delta \left( l - \frac{6n}{\pi^2} \right). \quad (2.14)$$

where the  $\delta$ -function comes from the integration over  $w$ , and the exponential factor is the ordinary degeneracy of bosonic string states in four-dimensional spacetime.

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<sup>2</sup> Here  $d = 2$  is the dimension of physical states, i.e. the transverse oscillators are  $N_m^i, i = 1, 2$ .

### 3. Energy Density of Cosmic Strings

In this section we will use the steepest descent contour method developed in [7][10] to estimate the energy density of strings produced during reheating. In [8], the quantization of bosonic strings has been done in a de Sitter background. The on-shell constraint for quantum states of a string leads to an equation of the form:

$$\left\{ \partial_t^2 + 3H\partial_t + k^2 e^{-2Ht} + 4N + 2E_0 - \sum_{m,i} H^2 \left( 1 + 2N_m^i \tilde{N}_m^i + N_m^i + \tilde{N}_m^i \right) \right\} \phi(N_m^i, \tilde{N}_m^i, \omega, k^i) = 0, \quad (3.1)$$

where we take  $\alpha' = 1$ .  $H$  is the Hubble parameter,  $N_m^i$ ,  $\tilde{N}_m^i$  are occupation number operators and  $N \equiv \sum_{i,m} m(N_m^i + \tilde{N}_m^i)$ ,  $k^i$  is the momentum vector in the four-dimensional spacetime and  $E_0$  is the center-of-mass energy. A general physical state  $|\phi\rangle$  corresponding to the string modes is related to  $\phi(N_m^i, \tilde{N}_m^i, \omega, k^i)$  as follows,

$$|\phi\rangle = \sum_{N_m^i, \tilde{N}_m^i} |N_m^i, \tilde{N}_m^i, \omega, k^i\rangle \phi(N_m^i, \tilde{N}_m^i, \omega, k^i), \quad (3.2)$$

where the definition of  $|N_m^i, \tilde{N}_m^i, \omega, k^i\rangle$  can be found in [8]. Eq.(3.1) is regarded as the equation of motion for the field of the corresponding string state. As we mentioned in the introduction, this equation is different from that used in [7], this difference makes our new result different from others.

There is no string production in pure de Sitter space with a constant Hubble parameter, even with the modified string spectrum as in (3.1). The only chance for string production to occur is the short period of reheating during which  $H$  becomes time-dependent. Thus we have to make a step forward, i.e. to generalize the equation of motion (3.1) to the case when  $H$  varies with time. This is developed in Appendix B. From now on  $(k^i)^2 e^{-2Ht}$  in (3.1) is replaced by  $(k^i)^2/a(t)^2$ , where  $a(t)$  is the cosmological scale factor.

It is convenient to introduce  $\phi(t)$  via  $\Phi(t) \equiv a\phi(t)$ , thus the equation of motion for  $\phi(t)$  is:

$$\ddot{\phi} + H\dot{\phi} + \left[ \left( \frac{k^i}{a(t)} \right)^2 + 4N + 2E_0 - C^2 H^2 \right] \phi = 0 \quad (3.3)$$

where

$$C^2 \equiv \sum_{m,i} \left( 3 - \epsilon + N_m^i + \tilde{N}_m^i + 2N_m^i \tilde{N}_m^i \right) \quad (3.4)$$

and

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \quad (3.5)$$

defined as the so-called slow-roll parameter, which is roughly equal to 1 at the end of inflation, and the dot denotes the derivative with respect to  $t$ . The slow-roll parameter is not larger than 2 in the case under study, see Appendix A.

In conformal time  $\eta$  defined by  $ad\eta = dt$ , we can eliminate the first order derivative term, and we will use prime to denote the derivative with respect to the conformal time. Thus (3.3) becomes:

$$\phi(\eta)'' + W(\eta)^2 \phi(\eta) = 0, \quad (3.6)$$

where

$$W(\eta)^2 \equiv k^2 + (4N + 2E_0)a(\eta)^2 - C^2 a(\eta)^2 H(\eta)^2. \quad (3.7)$$

Having obtained the equation of motion (3.6), now we use the steepest descent method to extract the approximate string pair production rate from (3.6). The steepest descent method was developed by various authors, especially, it was used to estimate the string production rate by Gubser [7]. The key assumption is that the occupation number  $|\beta|^2$  for a given mode is always much less than 1, where  $\beta$  is the Bogliubov coefficient. Setting

$$\phi(\eta) = \frac{\alpha(\eta)}{\sqrt{2W(\eta)}} e^{-i \int^\eta du W(u)} + \frac{\beta(\eta)}{\sqrt{2W(\eta)}} e^{i \int^\eta du W(u)}, \quad (3.8)$$

with the requirement  $|\alpha(\eta)|^2 - |\beta(\eta)|^2 = 1$ , we recast the equation (3.6) into

$$\alpha'(\eta) = \frac{W'}{2W} e^{2i \int^\eta du W(u)} \beta(\eta) \quad \beta'(\eta) = \frac{W'}{2W} e^{-2i \int^\eta du W(u)} \alpha(\eta). \quad (3.9)$$

Using the assumption  $\beta(\eta) \ll 1$  and  $\alpha(\eta) \approx 1$ , we obtain an approximate formula for  $\beta$

$$\beta \approx \int_{-\infty}^{\infty} d\eta \frac{W'}{2W} \exp \left( -2i \int^\eta du W(u) \right). \quad (3.10)$$

The integral in the exponential of (3.10) can be calculated as follows

$$\int_{\eta_i}^{\eta} du W(u) = \left[ \int_{\eta_i}^{\eta^*} + \int_{\eta^*}^{\eta} \right] du W(u) \quad (3.11)$$

where  $\eta_i$  is some initial time and  $\eta^*$  is defined to make  $W(\eta^*) = 0$ . Here the second term on the right hand side of (3.11) can be calculated as follows:

$$\begin{aligned} \int_{\eta^*}^{\eta} du W(u) &= \int_{\eta^*}^{\eta} du \sqrt{k^2 + (4N + 2E_0)a(\eta)^2 - C^2 a(\eta)^2 H(\eta)^2} \\ &\approx \int_{\eta^*}^{\eta} du \sqrt{[(4N + 2E_0)2aa' - C^2 2aa'H^2 - C^2 a^2 2HH'](u - \eta^*)} \\ &= \frac{2}{3} \delta^{3/2} \sqrt{(4N + 2E_0)2aa' - C^2 2aa'H^2 - C^2 a^2 2HH'} \end{aligned} \quad (3.12)$$

where we have expanded terms in the square root around  $\eta^*$ , and defined  $\delta \equiv \eta - \eta^*$ . Thus

$$\beta \approx I_0 \exp(-2i \int_{\eta_i}^{\eta^*} du W(u)) \quad (3.13)$$

where

$$I_0 \equiv \int_{-\infty}^{\infty} d\eta \frac{W'}{2W} \exp(-\frac{4i}{3} \delta^{3/2} \sqrt{S(\eta^*)}) \quad (3.14)$$

and

$$S(\eta^*) = (4N + 2E_0)2aa' - C^2 2aa'H^2 - C^2 a^2 2HH'. \quad (3.15)$$

Expanding  $W(\eta)$  around  $\eta^*$ , we get

$$I_0 = \frac{1}{4} \int_{-\infty}^{\infty} \frac{d\delta}{\delta} \exp\left(-\frac{4i}{3} \delta^{3/2} \sqrt{S(\eta^*)}\right). \quad (3.16)$$

From [10] we know that integrals such as (3.16) can be calculated by the contour integral method, and the result for (3.16) is simply  $I_0 = i\pi/3$ .

Now (3.13) can be written as:

$$\beta \approx \frac{i\pi}{3} \exp\left(-2i \int_{\eta_i}^r du W(u)\right) \exp\left(-2i \int_r^{\eta^*} du W(u)\right) \quad (3.17)$$

where  $r$  is the real part of  $\eta^* \equiv r - iu$ , and  $r, u$  are real with  $u > 0$ . Since what we need is the modulus of  $\beta$ , the main contribution comes from the second exponential function in (3.17) whose argument is the following integral and can be expanded as:

$$\int_r^{\eta^*} du W(u) = W(r)(-iu) + W'(r) \frac{(-iu)^2}{2} + W''(r) \frac{(-iu)^3}{6} + \dots \quad (3.18)$$



As long as  $|\frac{W''}{W}| \ll |\frac{6}{u^2}|$ , we can truncate this expansion to the first term (even terms do not contribute to the modulus of  $\beta$  since they are real), and indeed in the following calculation one will see that  $|\frac{W''}{W}| \ll |\frac{6}{u^2}|$  is satisfied in our case. Thus we get

$$|\beta|^2 \approx \left(\frac{\pi}{3}\right)^2 \exp(-4uW(r)). \quad (3.19)$$

Here the imaginary part of  $\eta^\star$  can be derived by using (3.7) and expanding  $W(\eta^\star)$  around  $r$  as follows<sup>3</sup>

$$\begin{aligned} 0 \equiv W^2(\eta^\star) &= W^2(r) + 2W(r)W'(r)(-iu) + [W(r)W''(r) + W'^2(r)](-iu)^2 \\ &+ [W'''(r)W(r) + W'(r)W''(r) + 2W'(r)W''(r)] \frac{(-iu)^3}{3} \end{aligned} \quad (3.20)$$

We solve these equations as follows

$$\begin{aligned} W^2(r) - [W(r)W''(r) + W'^2(r)]u^2 &= 0 \\ 6W(r)W'(r) - [W'''(r)W(r) + 3W''(r)W'(r)]u^2 &= 0 \end{aligned} \quad (3.21)$$

Then

$$|\beta|^2 \approx \left(\frac{\pi}{3}\right)^2 \exp\left(\frac{-4W(r)}{\sqrt{W''(r)/W(r) + (W'(r)/W(r))^2}}\right). \quad (3.22)$$

In our case  $W(\eta)$  is expressed in (3.7), thus we get

$$\frac{W'}{W} = \frac{a'}{a} - \left(k^2\left(\frac{a'}{a}\right) + C^2\left[\frac{a''}{a} - 2\left(\frac{a'}{a}\right)^2\right]\right)/W^2 \approx \frac{a'}{a}, \quad (3.23)$$

and

$$\frac{W''}{W} = \left(\frac{W'}{W}\right)' + \left(\frac{W'}{W}\right)^2 \approx \frac{a''}{a}. \quad (3.24)$$

Thus we write approximately:

$$|\beta_k(n, l)|^2 \approx \exp\left\{\frac{-4(k^2/a_r^2 + 4n - a_r^2 H_r^2 l)}{\sqrt{4n(H_r^2 + R_r/6)}}\right\}, \quad (3.25)$$

where we wrote  $\beta$  as function of comoving momentum  $k$  and excitation modes  $n, l$  explicitly, where  $n$  and  $l$  are eigenvalues of operators in (2.1) respectively. We have dropped the

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<sup>3</sup> We truncate the expansion up to the fourth term because one can check that the next order terms are much smaller than these terms. And one can see that the condition  $|\frac{W''}{W}| \ll |\frac{6}{u^2}|$  is also satisfied.

factor of  $(\pi/3)^2$ ,  $H_r$  and  $R_r$  correspond to the Hubble expansion rate and Ricci scalar for the metric  $ds^2 = a(\eta)^2(d\eta^2 - dx^{i^2})$  respectively. We emphasize that all time-dependent quantities in (3.25) are evaluated at  $\eta = r$ , where  $r$  is the real part of  $\eta^*$  given by  $W(\eta^*) = 0$ . From (3.25) we can see that indeed the production of highly-excited strings i.e. strings with large  $n$  and  $l$  are exponentially suppressed.

The total energy density of strings produced may be written as

$$\rho(\eta) = \frac{1}{2\pi^2 a(\eta)^3} \int dk k^2 \sum_{n,l} D_{n,l} |\beta_k(n,l)|^2 M_{n,l}(\eta), \quad (3.26)$$

where  $k$  is the comoving momentum,  $M_{n,l}(\eta)$  is the energy of a single string with excitation modes  $(n,l)$ , given by

$$\begin{aligned} M_{n,l}(\eta)^2 &= 4N + 2E_0 - \sum_{m,i} H^2 (1 + 2N_m^i \tilde{N}_m^i + N_m^i + \tilde{N}_m^i) \\ &= 4n + 2E_0 - H^2(1 + l) \\ &\approx 4n - H^2 l \end{aligned} \quad (3.27)$$

Now insert (3.25) and (3.27) into (3.26), we obtain approximate formulas:

$$\begin{aligned} \rho(\eta) &\approx \frac{1}{2\pi a^3} \int dk k^2 \sum_{n,l} n^{-\frac{5}{4}} \exp\left(\frac{2\pi}{3}\sqrt{6n}\right) \delta\left(l - \frac{6n}{\pi^2}\right) \times \\ &\quad \exp\left(-4 \frac{k^2/a_r^2 + 4n - a_r^2 H_r^2 l}{\sqrt{4n(H_r^2 + R_r/6)}}\right) \sqrt{4n - H^2 l} \\ &\sim \frac{1}{a^3} \int dk k^2 \sum_n n^{-\frac{5}{4}} \exp\left(\frac{2\pi}{3}\sqrt{6n}\right) \times \\ &\quad \exp\left(-4 \frac{k^2/a_r^2 + 4n - a_r^2 H_r^2 \frac{6n}{\pi^2}}{\sqrt{4n(H_r^2 + R_r/6)}}\right) \sqrt{4n - H^2 \frac{6n}{\pi^2}} \end{aligned} \quad (3.28)$$

For highly excited string states, neglecting  $k$  is a good approximation, though not a uniform one if  $a(\eta)$  becomes arbitrary small in the past. We get:

$$\rho(\eta) \sim \frac{1}{a^3} \int dk k^2 \int_1^{n_{max}} dn n^{-\frac{3}{4}} \exp\left\{-\left(\frac{8 - 12a_r^2 H_r^2/\pi^2}{\sqrt{H_r^2 + R_r/6}} - \frac{2\sqrt{6}\pi}{3}\right) \sqrt{n}\right\}. \quad (3.29)$$

where we have dropped the constant factor. We define  $A = \left(\frac{8 - 12a_r^2 H_r^2/\pi^2}{\sqrt{H_r^2 + R_r/6}} - \frac{2\sqrt{6}\pi}{3}\right)$  for short, the above integral can be approximated as:

$$\rho(\eta) \sim \frac{1}{a^3 A} \int dk k^2 \left( \text{Erf}\left(\sqrt{A} n^{\frac{1}{4}}_{max}\right) - \text{Erf}\left(\sqrt{A}\right) \right), \quad (3.30)$$

where  $Erf(x)$  is the error function. For very large  $x$ , approximately we have

$$Erf(x) \approx 1 - \frac{e^{-x^2}}{x\sqrt{\pi}} \left[ 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2x^2)^n} \right] \sim 1 - \frac{e^{-x^2}}{x\sqrt{\pi}}. \quad (3.31)$$

Thus we have

$$\rho(\eta) \sim \frac{1}{a(\eta)^3 A^{3/2}} \left( e^{-A} - \frac{e^{-A\sqrt{n_{max}}}}{n_{max}^{1/4}} \right), \quad (3.32)$$

where  $A$  is given by  $A = \left( \frac{8-12a_r^2 H_r^2/\pi^2}{\sqrt{H_r^2 + R_r/6}} - \frac{2\sqrt{6}\pi}{3} \right)$ . Since we consider the highly excited strings, i.e. strings with small momentum  $k$ , the integral in (3.30) with respect to the comoving momentum  $k$  will contribute a small factor which we have dropped. This result will not affect the qualitative behavior of the production rate of strings with respect to the string excitation modes  $n$  and  $l$ . The upper limit of  $n$  is roughly  $n_{max} \sim H^{-2}/4 \gg 1$  (See Appendix B), thus  $\rho \sim \frac{e^{-A}}{a(\eta)^3 A^{3/2}}$ . We emphasize that indeed  $H_r \ll \frac{1}{\sqrt{\alpha'}}$ . Thus in unit where  $\alpha' = 1$ ,  $H_r \ll 1$  and  $A \gg 1$ , the above approximation is qualitatively correct. Especially, since  $A \gg 1$ , from (3.32) we can see that the energy density of strings produced is very small and exponentially suppressed, i.e., highly excited strings are hardly produced in our case.

#### 4. Discussion

We have estimated the energy density of strings produced at the end of inflation and during reheating, our main result is

$$\rho(\eta) \sim \frac{\alpha'^{-2}}{a(\eta)^3 A^{3/2}} \left( e^{-A} - \frac{e^{-A\sqrt{n_{max}}}}{n_{max}^{1/4}} \right), \quad (4.1)$$

here  $A$  is given by  $A = \left( \frac{8-12a_r^2 H_r^2 \alpha'/\pi^2}{\sqrt{H_r^2 \alpha' + R_r \alpha'/6}} - \frac{2\sqrt{6}\pi}{3} \right)$ . We have reinstated  $\alpha'$  which has been set to 1 in our paper. Although it is difficult to get the explicit form of  $H_r$  due to the complicated equations of (3.21), one can make sure that  $H_r \sqrt{\alpha'}$  must be much smaller than 1, i.e. the Hubble scale in string production process is much lower than the string scale. In other words, the curvature radius is much larger than the string length, and a spacetime description gives a good approximation. In the case of a small  $H_r$ , due to the large exponential factor in (4.1), one can see that the energy density of strings produced is exponentially suppressed and indeed highly excited strings are hardly produced. (3.7).

Planck scale, otherwise the effective field theory viewpoint we used will be broken down, so that the energy density is also small.

In conclusion, we have shown in this paper that highly excited strings are hardly produced at the end of inflation, because  $|\beta|^2$  is highly suppressed by an exponential factor within and the degeneracy of highly excited strings is not sufficiently large to compensate it, thus the energy density is also suppressed by this exponential factor.

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### Appendix A. Hubble Parameter During Reheating

In most of popular inflation scenarios, the temperature is practically zero during inflation, relativistic matter is produced during the short reheating period when the inflaton oscillates coherently and decays to matter. Generally it is not known how the inflaton is coupled to a generic string state, so the usual reheating mechanism is not easily applied to the production of strings.

However, the spacetime metric is also coupled to strings, the details of the coupling can be seen from the string spectrum directly. When  $H$  remains nearly a constant, there is no string production. During the reheating period, the Hubble parameter is no longer a constant, and can be estimated by solving the Friedmann equation. In solving this equation, we should also take radiation into account. A more rigorous treatment should also take strings produced in the process into account, however, we do not know how to compute string energy density as a function of time (to this end, it is required to compute the string production rate per unit time). The Friedmann equation reads

$$3H^2 = \rho_\phi + \rho_\gamma, \quad (\text{A.1})$$

where we set  $8\pi G = 1$ , and  $\rho_\phi(t)$  and  $\rho_\gamma(t)$  are the energy densities of the inflaton and radiation respectively, their equations of motion are

$$\dot{\rho}_\phi + 3H\rho_\phi + \Gamma\rho_\phi = 0, \quad (\text{A.2})$$

and

$$\dot{\rho}_\gamma + 4H\rho_\gamma - \Gamma\rho_\phi = 0, \quad (\text{A.3})$$

where  $\Gamma$  is the decay rate of the inflaton and dot denotes the derivative with respect to the comoving time  $t$ .

Taking derivative of (A.1) with respect to  $t$  and using (A.2) (A.3) to eliminate  $\dot{\rho}_\phi$  and  $\dot{\rho}_\gamma$ , and then using (A.1) again, we find

$$\rho_\gamma = \frac{2\epsilon - 3}{4 - 2\epsilon} \rho_\phi, \quad (\text{A.4})$$

where  $\epsilon \equiv -\dot{H}/H^2$ . Now the Friedmann equation is

$$3H^2 = \frac{1}{4 - 2\epsilon} \rho_\phi. \quad (\text{A.5})$$

Combining (A.4) and (A.5), we get  $1.5 < \epsilon < 2$ . This result can be generalized when there are more energy components in the Universe.

We solve (A.2) in the limit<sup>4</sup> of  $\Gamma \gg H$ , which means that the inflaton decays very fast, indeed this is the case during reheating. Thus (A.2) is simplified to

$$\dot{\rho}_\phi + \Gamma \rho_\phi = 0. \quad (\text{A.6})$$

The general solution is

$$\rho_\phi = \rho_0 \exp(-\Gamma t), \quad (\text{A.7})$$

where  $\rho_0$  is an integration constant. Using solution (A.7), (A.5) can be rewritten as a differential equation of  $H$  explicitly

$$6\dot{H} + 12H^2 - \rho_0 \exp(-\Gamma t) = 0. \quad (\text{A.8})$$

This equation can be cast into the standard form of Bessel equation by changing variable  $t$  into  $\tau = \exp(-\Gamma t/2)$ ,

$$\frac{d^2(a^2)}{d\tau^2} + \frac{1}{\tau} \frac{d(a^2)}{d\tau} - \frac{4\rho_0}{3\Gamma^2} (a^2) = 0. \quad (\text{A.9})$$

The general solution to this equation is a linear combination of the modified Bessel functions of the first kind  $I_0$  and of the second kind  $K_0$ :

$$a^2 = c_1 I_0 \left( \sqrt{\frac{\rho_0}{3}} \frac{2\tau}{\Gamma} \right) + c_2 K_0 \left( \sqrt{\frac{\rho_0}{3}} \frac{2\tau}{\Gamma} \right), \quad (\text{A.10})$$

where  $c_1$  and  $c_2$  are integration constants.

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<sup>4</sup> Of course one can exactly solve the equation, but that is not necessary.

## Appendix B. Constraint Equation with Arbitrary $H(t)$

The equation of motion of bosonic string states derived in [8] is valid in pure de Sitter space, of which the Hubble parameter is a constant. In order to study string production, we focus on the reheating phase, where the Hubble parameter is varying with time, the spectrum formula of [8] cannot be used directly for our purpose. In this appendix we generalize the original result of [8] to the case of arbitrary  $H(t) \equiv \dot{a}(t)/a(t)$ . We refer the readers to the original paper [8] for more details of deriving the equation of motion when  $H$  is a constant.

The key point of the generalization is to re-calculate  $c_m$  in (3.16) of [8] in arbitrary  $H(t)$  case. The general definition of  $c_m$  is

$$c_m = e^{i(\phi_m - \psi_m)} \left\{ \alpha_m^* \dot{\beta}_m^* - \beta_m^* \dot{\alpha}_m^* + \left[ \alpha^{*2} e^{-2i \int^t du \lambda_m(u)} - \beta^{*2} e^{+2i \int^t du \lambda_m(u)} \right] \frac{\dot{\lambda}_m}{2\lambda_m} \right\}, \quad (\text{B.1})$$

where the most general form of  $\alpha_m$  and  $\beta_m$  are given by (3.7) and (3.8) of [8],

$$\begin{aligned} \alpha_m &= \cosh(\gamma_m) e^{i\delta_m + i\phi_m}, & \tilde{\alpha}_m &= \cosh(\gamma_m) e^{i\delta_m + i\psi_m}, \\ \beta_m &= \sinh(\gamma_m) e^{i\phi_m}, & \tilde{\beta}_m &= \sinh(\gamma_m) e^{i\psi_m}. \end{aligned} \quad (\text{B.2})$$

$\lambda_m$  in (B.1) is defined as

$$\lambda_m \equiv \text{sgn}(m) \sqrt{\frac{m^2}{\omega^2} - \eta \partial_t^2 \eta^{-1}}, \quad (\text{B.3})$$

where  $\eta \equiv \frac{1}{e^{Ht} \sqrt{\omega}}$  as in [8], and should be replaced by a formula in which  $\exp(Ht)$  is replaced by  $a(t)$ , and  $\gamma_m$  and  $\delta_m$  in (B.2) can be solved directly from (3.9-3.10) of [8].

$$\gamma_m = \cosh^{-1} \sqrt{\frac{\omega}{4m\lambda_m} \left[ \Gamma^2 + \lambda_m^2 + \frac{m^2}{\omega^2} \right] + \frac{1}{2}} \quad (\text{B.4})$$

$$\delta_m = \arctan \left( \frac{-2\lambda_m \Gamma}{2\Gamma^2 - \dot{\Gamma}} \right) - 2 \int^t du \lambda_m(u). \quad (\text{B.5})$$

here we define  $\Gamma \equiv \dot{\eta}/\eta$  for short. Insert (B.2) into (B.1) and use (B.3)-(B.5), after a tedious calculation we finally get

$$c_m = e^{-i(\delta_m + \psi_m + \phi_m)} \left( \frac{\dot{a}}{a} \right) \frac{\left[ \frac{\ddot{a}}{a} + \partial_t \left( \frac{\dot{\omega}}{2\omega} \right) - \left( \frac{\dot{a}}{a} \right)^2 \right] - i \frac{2m}{\omega} \left( \frac{\dot{a}}{a} + \frac{\dot{\omega}}{2\omega} \right)}{\sqrt{\left[ \frac{\ddot{a}}{a} + \partial_t \left( \frac{\dot{\omega}}{2\omega} \right) - \left( \frac{\dot{a}}{a} \right)^2 \right] + \frac{4m^2}{\omega^2} \left( \frac{\dot{a}}{a} + \frac{\dot{\omega}}{2\omega} \right)}}. \quad (\text{B.6})$$

What is needed in the calculation of the main context is the modulus of  $c_m$

$$|c_m| = \frac{\dot{a}}{a} \equiv H(t), \quad (\text{B.7})$$

it agrees with the original result of [8] where  $H$  is a constant. (B.7) tells us that the equation of motion of string states in the reheating phase where Hubble parameter is varying with time  $H = H(t)$  is simply given by the equation of motion of [8] when  $H$  is replaced by  $H(t) = \dot{a}(t)/a(t)$ .

At the end of this appendix we want to recall that the real condition of each  $\lambda_m$  is

$$\frac{m^2}{\omega^2} - (H + \frac{\dot{\omega}}{2\omega})^2 - \partial_t(\frac{\dot{\omega}}{2\omega}) > 0 \quad (\text{B.8})$$

where  $\omega^2 = 4N + 2E_0 + (p^i/a)^2$ . In fact if  $\lambda_1$  is real, so are  $\lambda_m$  for  $m > 1$ . Then the condition becomes

$$\frac{1}{\omega^2} - \left[ 1 + \frac{\epsilon}{2} \left( \frac{p^i}{a\omega} \right)^2 - \frac{3}{4} \left( \frac{p^i}{a\omega} \right)^4 \right] H^2 > 0. \quad (\text{B.9})$$

Because

$$\left( \frac{p}{a\omega} \right)^2 = \frac{(p^i/a)^2}{4N + 2E_0 + (p^i/a)^2} \ll 1 \quad (\text{B.10})$$

and  $\epsilon \in (1.5, 2)$ , the condition (B.9) is approximated by

$$\frac{1}{\omega^2} - \left[ 1 + \frac{\epsilon}{2} \left( \frac{p^i}{a\omega} \right)^2 \right] H^2 > 0. \quad (\text{B.11})$$

Thus we get the upper limit of  $n$  as

$$n < n_{max} = \frac{1}{4} \left( H^{-2} - \left( 1 + \frac{\epsilon}{2} \right) k^2 - 2E_0 \right). \quad (\text{B.12})$$

where  $k^i = p^i/a$  is the physical momentum.

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